Journal of Statistical Physics, Vol. 104, Nos. 3/4, 2001

Coulomb Systems with Ideal Dielectric Boundaries: Free Fermion Point and Universality

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Received December 20, 2000; revised February 19, 2001

A two-component Coulomb gas confined by walls made of ideal dielectric material is considered. In two dimensions at the special inverse temperature $\beta = 2$, by using the Pfaffian method, the system is mapped onto a four-component Fermi field theory with specific boundary conditions. The exact solution is presented for a semi-infinite geometry of the dielectric wall (the density profiles, the correlation functions) and for the strip geometry (the surface tension, a finite-size correction of the grand potential). The universal finite-size correction of the grand potential is shown to be a consequence of the good screening properties, and its generalization is derived for the conducting Coulomb gas confined in a slab of arbitrary dimension $\geqslant 2$ at any temperature.

KEY WORDS: Coulomb systems; solvable models; surface tension; correlation functions; finite-size effects; universality.

1. INTRODUCTION

Very recently, the interface between a Coulomb gas and a dielectric wall with zero dielectric constant (termed ideal dielectric) has been studied by one of us (L.Š.)[(1)] in the case of an exactly solvable two-dimensional model[(2)]: a two-component plasma of point-particles. The surface tension has been obtained, for an arbitrary temperature within the range of surface stability of the model $\beta < 3$ (β is the inverse temperature). This range exceeds the bulk range of stability $\beta < 2$. However, the method used in ref. 1, a mapping onto a solvable boundary sine-Gordon field theory, provided only the surface tension but did not give any information about the

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0022-4715/01/0800-0753$19.50/0

microscopic structure of the interface: density profiles and correlation functions.

It can be expected that this microscopic information is obtainable at the special inverse temperature $\beta = 2$. This is because, at this special temperature, for solving other problems, the model can be mapped onto a free fermion field theory (This temperature is the boundary of the stability domain of the model: for point particles, at a given fugacity, the bulk density and other bulk thermodynamic quantities diverge, however they can be made finite by the introduction of a small hard core in the interaction. Anyhow the *truncated* many-body densities are finite, even for point-particles). The two-dimensional two-component plasma at $\beta = 2$ has been extensively studied, in the bulk,[3] and also near a variety of possible interfaces, in particular the interface with a plain (simple) hard wall[4] and with an ideal conductor wall.[4–7] However, the case of the interface with an ideal dielectric could not be solved by the simplest possible extension of the mapping on free fermions with an inhomogeneous "mass" term which had been used for other interfaces (for the one-component plasma at $\beta = 2$, the exact solution in the case of an ideal dielectric wall was derived in refs. 8 and 9).

In the present paper, we do solve the problem of the interface with an ideal dielectric wall for the two-component plasma at $\beta = 2$ by an appropriate generalization of the mapping. In the previously studied cases, the mapping was obtained by associating a two-component Fermi field with each point of space. In the present case, inspired by remotely related ref. 10, we realized that a mapping on a four-component Fermi field is needed. Through that mapping, we are able to compute the density profile and the correlation functions near the interface. This is described in Section 2 dealing with a semi-infinite geometry of the dielectric wall.

In Section 3, we turn to the problem of the two-dimensional model at $\beta = 2$ confined in a strip of width W between two ideal dielectric walls. We show that the grand potential has the same universal finite-size correction of order $1/W$ as in the previously studied case of ideal conductor walls.[7] Actually, as shown in Section 4, this finite-size correction is a consequence of the good screening properties and its generalization can be derived for a conducting Coulomb system of arbitrary dimension at any temperature.

2. MODEL AND METHOD OF SOLUTION

2.1. Pfaffian Method

We consider an infinite 2D space of points $\boldsymbol{r} \in R^2$ defined by Cartesian (x, y) or complex $(z = x + \mathrm{i}y, \bar{z} = x - \mathrm{i}y)$ coordinates. The model dielectric-electrolyte interface is localized along the x axis at $y=0$. The half-space $y<0$ is assumed to be occupied by an ideal dielectric wall of dielectric constant $\epsilon=0$, impenetrable to particles. The electrolyte in the complementary half-space $y>0$ is modeled by the classical 2D two-component plasma of point particles $\{j\}$ of charge $\{q_j=\pm 1\}$, immersed in a homogeneous medium of dielectric constant $=1$ and interacting via Coulomb interaction.

At the special inverse temperature $\beta=2$, in order to avoid the collapse of positive and negative particles, we start with a lattice version of the Coulomb model. There are two interwoven sublattices of sites $u\in U$ and $v\in V$. The positive (negative) particles sit on the sublattice U (V); each site can be either empty or occupied by one particle. We work in the grand canonical ensemble, with position-dependent fugacities $\zeta(u)$ and $\zeta(v)$ in order to generate multi-particle densities. In infinite space, the Coulomb potential v_0 at spatial position $\boldsymbol{r}$, induced by a charge at the origin, is $v_0(\boldsymbol{r})=-\ln(|\boldsymbol{r}|/a)$, where the length constant a fixes the zero point of energy. The presence of a dielectric wall induces to a particle at $\boldsymbol{r}=(x,y)$, or z, an image at $\boldsymbol{r}^*=(x,-y)$, or $\bar{z}$, of the same charge.[11] We consider only neutral configurations of N positive and N negative particles, the complex coordinates of which are u_i and v_i, respectively. The corresponding Boltzmann interaction factor, involving direct particle-particle interactions $q_iq_jv_0(|\boldsymbol{r}_i-\boldsymbol{r}_j|)$ and interactions of particles with the images of other particles $q_iq_jv_0(|\boldsymbol{r}_i-\boldsymbol{r}_j^*|)$ as well as their self-images $(1/2)\,q_i^2v_0(|\boldsymbol{r}_i-\boldsymbol{r}_i^*|)$, is, at $\beta=2$,

$$\begin{aligned}
&B_N(\{u_i\};\{v_i\})\\
&\quad=(-1)^N a^{2N}\prod_i(u_i-\bar{u}_i)(v_i-\bar{v}_i)\\
&\qquad\times\frac{\prod_{i<j}(u_i-u_j)(\bar{u}_i-\bar{u}_j)(u_i-\bar{u}_j)(\bar{u}_i-u_j)(v_i-v_j)(\bar{v}_i-\bar{v}_j)(v_i-\bar{v}_j)(\bar{v}_i-v_j)}{\prod_{i,j}(u_i-v_j)(\bar{u}_i-\bar{v}_j)(u_i-\bar{v}_j)(\bar{u}_i-v_j)}
\end{aligned}\tag{2.1}$$

By using an identity of Cauchy[12]

$$\det\left(\frac{1}{z_i-z'_j}\right)_{i,j=1,\dots,2N}=(-1)^{N(2N-1)}\frac{\prod_{i<j}(z_i-z_j)(z'_i-z'_j)}{\prod_{i,j}(z_i-z'_j)}\tag{2.2a}$$

with the identification

$$z_{2i-1}=u_i,\qquad z_{2i}=\bar{u}_i,\qquad z'_{2i-1}=v_i,\qquad z'_{2i}=\bar{v}_i,\qquad\qquad i=1,\dots,N\tag{2.2b}$$

the Boltzmann factor (2.1) is expressible as the determinant of an $N \times N$ matrix whose elements are 2×2 matrices:

$$B_N(\{u_i\}; \{v_i\}) = \det \begin{pmatrix} \dfrac{a}{u_i - v_j} & \dfrac{a}{u_i - \bar{v}_j} \\ \dfrac{a}{\bar{u}_i - v_j} & \dfrac{a}{\bar{u}_i - \bar{v}_j} \end{pmatrix}_{i,j=1,\dots,N} . \tag{2.3}$$

The grand partition function reads

$$\begin{aligned} \Xi = 1 &+ \sum_{\substack{u \in U \\ v \in V}} \zeta(u)\, \zeta(v)\, B_1(u; v) \\ &+ \sum_{\substack{u_1 < u_2 \in U \\ v_1 < v_2 \in V}} \zeta(u_1)\, \zeta(u_2)\, \zeta(v_1)\, \zeta(v_2)\, B_2(u_1, u_2; v_1, v_2) + \cdots \end{aligned} \tag{2.4}$$

where the sums are defined in such a way that configurations which differ by a permutation of identical-charge particles are counted only once.

We introduce a couple of Grassmann variables for every site $u \in U$, (ψ_u^1, ψ_u^2), and every site $v \in V$, (ψ_v^1, ψ_v^2), and order all of them into a single vector ${}^t\theta = (\cdots \psi_u^1, \psi_u^2 \cdots \psi_v^1, \psi_v^2 \cdots)$ whose components obey the ordinary anticommutation rules $\{\theta_i, \theta_j\} = 0$. Let us consider the Grassmann integral

$$Z_0 = \int \mathrm{d}\theta\, \mathrm{e}^{-S_0} \tag{2.5a}$$

where $\mathrm{d}\theta = \prod_v \mathrm{d}\psi_v^2\, \mathrm{d}\psi_v^1 \prod_u \mathrm{d}\psi_u^2\, \mathrm{d}\psi_u^1$ and

$$-S_0 = \tfrac{1}{2}\, {}^t\theta \mathbf{A} \theta \tag{2.5b}$$

is (minus) the Gaussian action with an antisymmetric matrix $\mathbf{A}$, $A_{ij} + A_{ji} = 0$. We recall that $\mathbf{A}^{-1}$ is also antisymmetric. The averaging with S_0 will be denoted by

$$\langle \cdots \rangle = \frac{1}{Z_0} \int \mathrm{d}\theta \cdots \mathrm{e}^{-S_0} \tag{2.6}$$

Gaussian integrals of type (2.5) are expressible as[13]

$$Z_0 = \mathrm{Pf}(\mathbf{A}) \tag{2.7}$$

where Pf(**A**) is the Pfaffian of the antisymmetric matrix **A**, satisfying the well known identity

$$\mathrm{Pf}(\mathbf{A})^2 = \det \mathbf{A} \tag{2.8}$$

The two-variable averages are given by

$$\langle \theta_i \theta_j \rangle = (A^{-1})_{ji} \tag{2.9}$$

The standard Wick theorem for fermions generalizes to higher-order averages of θ variables.[14] The **A**-matrix will be chosen such that its inverse satisfies the following relations:

$$(A^{-1})^{\alpha\beta}_{uu'} = \langle \psi^{\beta}_{u'} \psi^{\alpha}_{u} \rangle = 0 \tag{2.10a}$$

$$(A^{-1})^{\alpha\beta}_{vv'} = \langle \psi^{\beta}_{v'} \psi^{\alpha}_{v} \rangle = 0 \tag{2.10b}$$

i.e., the sites of the same sublattice-type do not interact with each other, and

$$(A^{-1})^{\alpha\beta}_{uv} = \langle \psi^{\beta}_{v} \psi^{\alpha}_{u} \rangle = \begin{pmatrix} \dfrac{a}{u-v} & \dfrac{a}{u-\bar{v}} \\ \dfrac{a}{\bar{u}-v} & \dfrac{a}{\bar{u}-\bar{v}} \end{pmatrix} \tag{2.10c}$$

$$(A^{-1})^{\alpha\beta}_{vu} = \langle \psi^{\beta}_{u} \psi^{\alpha}_{v} \rangle = \begin{pmatrix} \dfrac{a}{v-u} & \dfrac{a}{v-\bar{u}} \\ \dfrac{a}{\bar{v}-u} & \dfrac{a}{\bar{v}-\bar{u}} \end{pmatrix} \tag{2.10d}$$

with the required antisymmetry property $(A^{-1})^{\alpha\beta}_{uv} = -(A^{-1})^{\beta\alpha}_{vu}$. Here, the rows and the columns of the matrices are numbered as $\alpha, \beta = 1, 2$.

We now introduce an antisymmetric "mass" matrix,

$$M^{\alpha\alpha'}_{uu'} = \delta_{uu'} \begin{pmatrix} 0 & \mathrm{i}\zeta(u) \\ -\mathrm{i}\zeta(u) & 0 \end{pmatrix} \tag{2.11a}$$

$$M^{\alpha\alpha'}_{vv'} = \delta_{vv'} \begin{pmatrix} 0 & \mathrm{i}\zeta(v) \\ -\mathrm{i}\zeta(v) & 0 \end{pmatrix} \tag{2.11b}$$

$$M^{\alpha\beta}_{uv} = 0 \tag{2.11c}$$

$$M^{\alpha\beta}_{vu} = 0 \tag{2.11d}$$

Define

$$Z=\int \mathrm{d}\theta\, \mathrm{e}^{-S} \tag{2.12a}$$

with the action

$$\begin{aligned}-S&=\tfrac{1}{2}\,{}^t\theta(\mathbf{A}+\mathbf{M})\,\theta\\ &=-S_0+\sum_u \mathrm{i}\zeta(u)\,\psi_u^1\psi_u^2+\sum_v \mathrm{i}\zeta(v)\,\psi_v^1\psi_v^2\end{aligned} \tag{2.12b}$$

Clearly,

$$Z=\mathrm{Pf}(\mathbf{A}+\mathbf{M}) \tag{2.13}$$

Let us expand the ratio Z/Z_0 in fugacities $\{\zeta(u)\}$ and $\{\zeta(v)\}$. Owing to (2.10), only neutral contributions with an equal number of $\{\zeta(u)\}$ and $\{\zeta(v)\}$ will survive,

$$\frac{Z}{Z_0}=1+\sum_{N=1}^{\infty}(-1)^N \sum_{\substack{u_1<u_2<\dots<u_N\in U\\ v_1<v_2<\dots<v_N\in V}} \prod_{i=1}^{N}\zeta(u_i)\,\zeta(v_i)\left\langle\prod_{i=1}^{N}(\psi_{u_i}^1\psi_{u_i}^2\psi_{v_i}^1\psi_{v_i}^2)\right\rangle \tag{2.14}$$

With the definition of pair averages (2.10), the Wick theorem for fermions (see, e.g., refs. 13 and 14) implies

$$\begin{aligned}\left\langle\prod_{i=1}^{N}(\psi_{u_i}^1\psi_{u_i}^2\psi_{v_i}^1\psi_{v_i}^2)\right\rangle&=(-1)^N\left\langle\prod_{i=1}^{N}(\psi_{v_i}^1\psi_{u_i}^1\psi_{v_i}^2\psi_{u_i}^2)\right\rangle\\ &=(-1)^N\det\begin{pmatrix}\dfrac{a}{u_i-v_j} & \dfrac{a}{u_i-\bar{v}_j}\\[2ex] \dfrac{a}{\bar{u}_i-v_j} & \dfrac{a}{\bar{u}_i-\bar{v}_j}\end{pmatrix}_{i,j=1,\dots,N}\end{aligned} \tag{2.15}$$

Comparing with (2.3) and (2.4) one concludes that

$$\Xi=\frac{Z}{Z_0}=\frac{\mathrm{Pf}(\mathbf{A}+\mathbf{M})}{\mathrm{Pf}(\mathbf{A})} \tag{2.16}$$

Introducing the matrix $\mathbf{K}=\mathbf{M}\mathbf{A}^{-1}$, (2.16) can be rewritten as

$$\ln\Xi=\tfrac{1}{2}\ln\det(1+\mathbf{K})=\tfrac{1}{2}\mathrm{Tr}\ln(1+\mathbf{K}) \tag{2.17}$$

Marking the charge sign of the particle at $r=(z,\bar{z})$ by an index $s=\pm$, i.e., identifying $u=(r,+)$, $v=(r,-)$ and $\zeta(u)=\zeta_+(r)$, $\zeta(v)=\zeta_-(r)$, the **K**-matrix has the elements

$$K_{ss'}^{\alpha\alpha'}(r,r')=\delta_{s,-s'}\,\mathrm{i}\zeta_s(r)\begin{pmatrix}\dfrac{a}{\bar{z}-z'} & \dfrac{a}{\bar{z}-\bar{z}'}\\[2ex] \dfrac{-a}{z-z'} & \dfrac{-a}{z-\bar{z}'}\end{pmatrix}\tag{2.18}$$

where the $\alpha, \alpha'=1, 2$ indices label the elements of the 2×2 matrix.

2.2. Many-Particle Densities

The calculation of many-body densities from the generator of type (2.17) is straightforward (see, e.g., ref. 3). When S is the area of a lattice cell, using the equality

$$\zeta_{s_1}(r_1)\frac{\partial}{\partial\zeta_{s_1}(r_1)}K_{ss'}^{\alpha\alpha'}(r,r')=\delta_{s_1,s}\delta_{r_1,r}K_{ss'}^{\alpha\alpha'}(r,r')\tag{2.19}$$

one finds for the density of particles of one sign (number of such particles per unit area)

$$\begin{aligned}n_{s_1}(r_1)&=\frac{1}{S}\zeta_{s_1}(r_1)\frac{\partial}{\partial\zeta_{s_1}(r_1)}\ln\Xi\Big|_{\zeta_s(r)=\zeta}\\&=\frac{1}{2}m\sum_{\alpha_1}G_{s_1s_1}^{\alpha_1\alpha_1}(r_1,r_1)\end{aligned}\tag{2.20}$$

Here, the matrix

$$\mathbf{G}=\frac{1}{2\pi a\zeta}\frac{\mathbf{K}}{1+\mathbf{K}}\tag{2.21}$$

is defined with K-elements evaluated at constant fugacity, $\zeta_s(r)=\zeta$, and the rescaled fugacity m is defined by $m=2\pi a\zeta/S$. Using

$$\frac{\partial}{\partial\zeta_{s_2}(r_2)}\frac{1}{1+\mathbf{K}}=-\frac{1}{1+\mathbf{K}}\frac{\partial\mathbf{K}}{\partial\zeta_{s_2}(r_2)}\frac{1}{1+\mathbf{K}}\tag{2.22}$$

one finds for the truncated two-body density

$$n^{(2)T}_{s_1s_2}(\boldsymbol{r}_1,\boldsymbol{r}_2)=\frac{1}{S^2}\zeta_{s_1}(\boldsymbol{r}_1)\,\zeta_{s_2}(\boldsymbol{r}_2)\,\frac{\partial^2}{\partial\zeta_{s_1}(\boldsymbol{r}_1)\partial\zeta_{s_2}(\boldsymbol{r}_2)}\ln\Xi\bigg|_{\zeta_s(\boldsymbol{r})=\zeta}$$

$$=-\frac{1}{2}m^2\sum_{\alpha_1,\alpha_2}G^{\alpha_1\alpha_2}_{s_1s_2}(\boldsymbol{r}_1,\boldsymbol{r}_2)\,G^{\alpha_2\alpha_1}_{s_2s_1}(\boldsymbol{r}_2,\boldsymbol{r}_1)\tag{2.23}$$

By successive iterations, one finds for the truncated k-body density

$$n^{(k)T}_{s_1\cdots s_k}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_k)=(-1)^{k+1}\tfrac{1}{2}m^k\sum_{\alpha_1,\ldots,\alpha_k}\sum_{(i_1i_2\cdots i_k)}G^{\alpha_{i_1}\alpha_{i_2}}_{s_{i_1}s_{i_2}}(\boldsymbol{r}_{i_1},\boldsymbol{r}_{i_2})\cdots G^{\alpha_{i_k}\alpha_{i_1}}_{s_{i_k}s_{i_1}}(\boldsymbol{r}_{i_k},\boldsymbol{r}_{i_1})\tag{2.24}$$

where the second summation runs over all cycles $(i_1i_2\cdots i_k)$ built with $\{1,2,\ldots,k\}$.

In the continuum limit where the lattice spacing goes to zero, the $\mathbf{G}_{ss'}$ $(s,s'=\pm)$ matrices, defined by (2.21), satisfy the integral equations

$$\mathbf{G}_{ss}(\boldsymbol{r}_1,\boldsymbol{r}_2)=-\mathrm{i}\left(\frac{m}{2\pi}\right)\int_D \mathrm{d}^2r\begin{pmatrix}\dfrac{1}{\bar z_1-z} & \dfrac{1}{\bar z_1-\bar z}\\ \dfrac{-1}{z_1-z} & \dfrac{-1}{z_1-\bar z}\end{pmatrix}\cdot\mathbf{G}_{-s,s}(\boldsymbol{r},\boldsymbol{r}_2)\tag{2.25a}$$

$$\mathbf{G}_{-s,s}(\boldsymbol{r}_1,\boldsymbol{r}_2)=\mathrm{i}\left(\frac{1}{2\pi}\right)\begin{pmatrix}\dfrac{1}{\bar z_1-z_2} & \dfrac{1}{\bar z_1-\bar z_2}\\ \dfrac{-1}{z_1-z_2} & \dfrac{-1}{z_1-\bar z_2}\end{pmatrix}$$

$$-\mathrm{i}\left(\frac{m}{2\pi}\right)\int_D \mathrm{d}^2r\begin{pmatrix}\dfrac{1}{\bar z_1-z} & \dfrac{1}{\bar z_1-\bar z}\\ \dfrac{-1}{z_1-z} & \dfrac{-1}{z_1-\bar z}\end{pmatrix}\cdot\mathbf{G}_{ss}(\boldsymbol{r},\boldsymbol{r}_2)\tag{2.25b}$$

where the domain of integration D is the half plane $y\geqslant 0$. The operators

$$\partial_z=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right),\qquad \partial_{\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right)\tag{2.26}$$

act as follows[13]

$$\partial_z \frac{1}{\bar{z}-\bar{z}'} = \pi\delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad \partial_{\bar{z}} \frac{1}{z-z'} = \pi\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{2.27}$$

Differentiating appropriately equations (2.25), one gets

$$\begin{pmatrix} 0 & \partial_{\bar{z}_1} \\ -\partial_{z_1} & 0 \end{pmatrix} \mathbf{G}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathrm{i}\left(\frac{m}{2}\right) \mathbf{G}_{-s,s}(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{2.28a}$$

$$\begin{pmatrix} 0 & \partial_{\bar{z}_1} \\ -\partial_{z_1} & 0 \end{pmatrix} \mathbf{G}_{-s,s}(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\mathrm{i}\frac{1}{2}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\,\mathbf{1} + \mathrm{i}\left(\frac{m}{2}\right) \mathbf{G}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{2.28b}$$

where $\mathbf{1}$ denotes the 2×2 unit matrix. Using $\partial_z \partial_{\bar{z}} = (1/4)\Delta$, these equations can be combined into

$$(-\Delta_1 + m^2)\,\mathbf{G}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) = m\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,\mathbf{1} \tag{2.29}$$

The boundary conditions are given by the integral equations (2.25):

$$G^{11}_{ss'}(\boldsymbol{r}_1, \boldsymbol{r}_2) + G^{21}_{ss'}(\boldsymbol{r}_1, \boldsymbol{r}_2) = 0, \qquad (\boldsymbol{r}_1 \vee \boldsymbol{r}_2) \in \partial D \tag{2.30a}$$

$$G^{12}_{ss'}(\boldsymbol{r}_1, \boldsymbol{r}_2) + G^{22}_{ss'}(\boldsymbol{r}_1, \boldsymbol{r}_2) = 0, \qquad (\boldsymbol{r}_1 \vee \boldsymbol{r}_2) \in \partial D \tag{2.30b}$$

where ∂D is the domain boundary $y = 0$.

Since $\mathbf{G}_{ss'}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is translationally invariant along the x axis, for solving the partial differential equations (2.28) with the boundary conditions (2.30), it is appropriate to introduce the Fourier transform with respect to $x_1 - x_2$ defined by

$$\mathbf{G}_{ss'}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}l}{2\pi}\, \tilde{\mathbf{G}}_{ss'}(y_1, y_2, l)\, \mathrm{e}^{il(x_1-x_2)} \tag{2.31}$$

We then obtain ordinary differential equations in y_1. Eq. (2.29) becomes

$$\left[-\frac{\partial^2}{\partial y_1^2} + \kappa^2 \right] \tilde{\mathbf{G}}_{ss}(y_1, y_2, l) = m\delta(y_1 - y_2)\,\mathbf{1} \tag{2.32}$$

where $\kappa = (m^2 + l^2)^{1/2}$, and we look for a solution of (2.32), symmetric in y_1 and y_2, in which a "reflected" part is added to the free space solution, i.e., of the form

$$\tilde{G}^{\alpha\alpha'}_{ss}(y_1, y_2, l) = \frac{m}{2\kappa}\left[\delta_{\alpha\alpha'}\mathrm{e}^{-\kappa|y_1-y_2|} + A^{\alpha\alpha'}(l)\,\mathrm{e}^{-\kappa(y_1+y_2)}\right] \tag{2.33}$$

where $A^{\alpha\alpha'}(l)$ are functions to be determined. The matrix elements $G^{\alpha\alpha'}_{-s,s}$ are related to (2.33) by the Fourier transforms of the relations (2.28a), and the Fourier transforms of the boundary conditions (2.30) determine the coefficients $A^{\alpha\alpha'}(l)$ in (2.33). The result is

$$\tilde{G}^{11}_{ss}(y_1, y_2, l) = \frac{m}{2\kappa}\left[\mathrm{e}^{-\kappa|y_1-y_2|} + \left(\frac{\kappa}{l} - 1\right)\mathrm{e}^{-\kappa(y_1+y_2)}\right] \tag{2.34a}$$

$$\tilde{G}^{22}_{ss}(y_1, y_2, l) = \frac{m}{2\kappa}\left[\mathrm{e}^{-\kappa|y_1-y_2|} - \left(\frac{\kappa}{l} + 1\right)\mathrm{e}^{-\kappa(y_1+y_2)}\right] \tag{2.34b}$$

$$\tilde{G}^{12}_{ss}(y_1, y_2, l) = \frac{m}{2l}\mathrm{e}^{-\kappa(y_1+y_2)} \tag{2.34c}$$

$$\tilde{G}^{21}_{ss}(y_1, y_2, l) = -\frac{m}{2l}\mathrm{e}^{-\kappa(y_1+y_2)} \tag{2.34d}$$

The inverse Fourier transforms are

$$G^{11}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{m}{2\pi}[K_0(mr_{12}) - K_0(mr^*_{12}) + \mathrm{i}I(x_1 - x_2, y_1 + y_2)] \tag{2.35a}$$

$$G^{22}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{m}{2\pi}[K_0(mr_{12}) - K_0(mr^*_{12}) - \mathrm{i}I(x_1 - x_2, y_1 + y_2)] \tag{2.35b}$$

$$G^{12}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathrm{i}\frac{m}{2\pi}I(x_1 - x_2, y_1 + y_2) \tag{2.35c}$$

$$G^{21}_{ss}(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\mathrm{i}\frac{m}{2\pi}I(x_1 - x_2, y_1 + y_2) \tag{2.35d}$$

where $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, $r^*_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}^*_2|$ with $\boldsymbol{r}^*_2 = (x_2, -y_2)$ the image of $\boldsymbol{r}_2$, K_0 is a modified Bessel function, and I is the function (obtained as a principal value)

$$I(x_1 - x_2, y_1 + y_2) = \int_0^\infty \mathrm{d}l\, \frac{\sin[l(x_1 - x_2)]}{l}\mathrm{e}^{-\kappa(y_1+y_2)} \tag{2.36}$$

Using the Fourier transforms of (2.28a) we obtain the $(-s, s)$ matrix elements

$$\tilde{G}^{11}_{-s,s}(y_1, y_2, l) = \frac{1}{2\kappa}\left(-\kappa + l + \frac{m^2}{l}\right) e^{-\kappa(y_1+y_2)} \tag{2.37a}$$

$$\tilde{G}^{22}_{-s,s}(y_1, y_2, l) = \frac{1}{2\kappa}\left(-\kappa - l - \frac{m^2}{l}\right) e^{-\kappa(y_1+y_2)} \tag{2.37b}$$

$$\tilde{G}^{12}_{-s,s}(y_1, y_2, l) = \frac{1}{2\kappa}\left([l - \kappa \operatorname{sign}(y_1 - y_2)]\, e^{-\kappa|y_1-y_2|} + \frac{m^2}{l}\, e^{-\kappa(y_1+y_2)}\right) \tag{2.37c}$$

$$\tilde{G}^{21}_{-s,s}(y_1, y_2, l) = \frac{1}{2\kappa}\left([-l - \kappa \operatorname{sign}(y_1 - y_2)]\, e^{-\kappa|y_1-y_2|} - \frac{m^2}{l}\, e^{-\kappa(y_1+y_2)}\right) \tag{2.37d}$$

The inverse Fourier transforms are

$$G^{11}_{-s,s}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi}\left[\frac{i(x_1 - x_2) - (y_1 + y_2)}{r^*_{12}} K_1(mr^*_{12}) + iJ(x_1 - x_2, y_1 + y_2)\right] \tag{2.38a}$$

$$G^{22}_{-s,s}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi}\left[\frac{-i(x_1 - x_2) - (y_1 + y_2)}{r^*_{12}} K_1(mr^*_{12}) - iJ(x_1 - x_2, y_1 + y_2)\right] \tag{2.38b}$$

$$G^{12}_{-s,s}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi}\left[\frac{i(x_1 - x_2) - (y_1 - y_2)}{r_{12}} K_1(mr_{12}) + iJ(x_1 - x_2, y_1 + y_2)\right] \tag{2.38c}$$

$$G^{21}_{-s,s}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi}\left[\frac{-i(x_1 - x_2) - (y_1 - y_2)}{r_{12}} K_1(mr_{12}) - iJ(x_1 - x_2, y_1 + y_2)\right] \tag{2.38d}$$

where K_1 is a modified Bessel function and J is the function

$$J(x_1 - x_2, y_1 + y_2) = \int_0^\infty dl\, \frac{m \sin[l(x_1 - x_2)]}{\kappa l}\, e^{-\kappa(y_1+y_2)} \tag{2.39}$$

Since $K_0(mr_{12})$ diverges logarithmically as $r_{12} \to 0$, using (2.35) in (2.20) gives divergent densities, like in the free space problem. However, if we subtract from $G^{\alpha\alpha}_{ss}$ its free space value, we get a finite result for the difference $n_s(y) - n_s(\infty)$ (a density depends only on y):

$$n_s(y) - n_s(\infty) = -\frac{m^2}{2\pi} K_0(2my) \tag{2.40}$$

Using (2.35) and (2.38) in (2.23), we get the truncated two-body densities

$$n_{ss}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^2}{2\pi}[K_0(mr_{12}) - K_0(mr_{12}^*)]^2 \tag{2.41a}$$

$$n_{-s,s}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m^2}{2\pi}\left\{[K_1(mr_{12})]^2 - [K_1(mr_{12}^*)]^2 + 2J(x_1 - x_2, y_1 + y_2)(x_1 - x_2)\left[\frac{K_1(mr_{12})}{r_{12}} - \frac{K_1(mr_{12}^*)}{r_{12}^*}\right]\right\} \tag{2.41b}$$

Writing (formally) the (infinite) bulk densities as $n_s(\infty) = (m^2/2\pi)\, K_0(0)$, we see on (2.40) that $n_s(0) = 0$, as expected because of the strong repulsion between a particle and its image. Similarly, one checks on (2.41) that also the two-body densities vanish, as expected, when one of the particles is on the boundary.

The correlation functions (2.41) are short-ranged. Indeed, they are expressed in terms of modified Bessel functions or related functions $[\partial J/\partial(x_1 - x_2) = mK_0(mr_{12}^*)]$ which have an exponential decay. This short-range behavior also occurs in the case of a one-component plasma[9] along an ideal dielectric wall. It is very likely that this is a generic feature. Indeed, when the dielectric constant of the wall is finite, a particle near the wall and its screening cloud of charge form a dipole perpendicular to the wall, which is responsible for the occurence of a long-range tail along the wall. However, in the case of a wall of zero dielectric constant, the dipole is cancelled by its electric image and the long-range tail does not occur.

As a consequence of this perfect screening, the charge correlation functions are expected to obey a variety of sum rules. In particular, an even multipole moment of a particle plus its surrounding screening cloud should vanish[15] (in the presence of an ideal dielectric boundary, the situation is different for odd multipole moments: some of their components trivially vanish for symmetry reasons, while other components have no a priori reason for vanishing since they are cancelled by the images). As a check of our results (2.40) and (2.41) for the one and two-body densities, we explicitly derive the monopole and quadrupole sum rules. Since the one-body densities are divergent, we subtract the corresponding bulk moment for the system in infinite space, without a boundary. Thus, we expect the monopole sum rule

$$\int_{y_1>0} d^2r_1 [n_{ss}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) - n_{s,-s}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) - n_{ss}^{(2)T}(r_{12}; \infty) - n_{ss}^{(2)T}(r_{12}^*; \infty) + n_{s,-s}^{(2)T}(r_{12}; \infty) + n_{s,-s}^{(2)T}(r_{12}^*; \infty)] = -n_s(y_2) + n_s(\infty) \tag{2.42}$$

and the quadrupole sum rule

$$\int_{y_1>0} d^2r_1 [y_1^2-(x_1-x_2)^2][n_{ss}^{(2)T}(r_1,r_2)-n_{s,-s}^{(2)T}(r_1,r_2)-n_{ss}^{(2)T}(r_{12};\infty)$$
$$-n_{ss}^{(2)T}(r_{12}^*;\infty)+n_{s,-s}^{(2)T}(r_{12};\infty)+n_{s,-s}^{(2)T}(r_{12}^*;\infty)]$$
$$=-y_2^2[n_s(y_2)-n_s(\infty)] \tag{2.43}$$

where $n_{ss}^{(2)T}(r_{12};\infty)=-(m^2/2\pi)^2\,[K_0(mr_{12})]^2$ and $n_{s,-s}^{(2)T}(r_{12};\infty)=(m^2/2\pi)^2\times[K_1(mr_{12})]^2$ are the bulk truncated two-body densities (the moments of the system without a boundary have been replaced by integrals over the half-space $y_1>0$ and the addition of $n_{ss'}^{(2)T}(r_{12}^*;\infty)$ terms). These sum rules are derived in the Appendix.

3. STRIP

We now consider a strip geometry. The Coulomb fluid, infinite in the x-direction, is constrained by two ideal dielectric walls at $y=0$ and $y=W$. The above studied half-plane case corresponds to the limit $W\to\infty$.

The Coulomb potential between the dielectric walls is the solution of the Poisson equation $\Delta v(\mathbf{r},\mathbf{r}')=-2\pi\delta(\mathbf{r}-\mathbf{r}')$ with the boundary conditions for the y-component of the electric field $E_y(x,0)=0$ and $E_y(x,W)=0$. By the method of images,[11] after subtracting an irrelevant infinite constant, one gets for the Coulomb potential

$$v(\mathbf{r},\mathbf{r}')=-\ln\left|\frac{\sinh k(z-z')}{ka}\frac{\sinh k(\bar{z}-z')}{ka}\right| \tag{3.1}$$

where $k=\pi/(2W)$. Each particle has a self-interaction $-(1/2)\ln|\sinh k(z-\bar{z})/(ka)|$. Writing

$$\sinh k(z-z')=\tfrac{1}{2}e^{-k(z+z')}(e^{2kz}-e^{2kz'}) \tag{3.2}$$

and using $\exp(2ku)$ and $\exp(2kv)$ instead of u and v, one can proceed as in the previous section, with the final result for the grand partition function, at $\beta=2$,

$$\ln \Xi=\tfrac{1}{2}\mathrm{Tr}\ln(1+\mathbf{K}) \tag{3.3}$$

where

$$K_{ss'}^{\alpha\alpha'}(\boldsymbol{r}, \boldsymbol{r}') = \delta_{s,-s'}\, \mathrm{i}\zeta \begin{pmatrix} \dfrac{ka}{\sinh k(\bar{z}-z')} & \dfrac{ka}{\sinh k(\bar{z}-\bar{z}')} \\ \dfrac{-ka}{\sinh k(z-z')} & \dfrac{-ka}{\sinh k(z-\bar{z}')} \end{pmatrix} \tag{3.4}$$

In the limit $W \to \infty$ $(k \to 0)$, (3.4) reduces to (2.18) as it should be.

In the continuum limit, keeping the previous definition of the rescaled fugacity $m = 2\pi a\zeta/S$, the eigenfunctions $\{\psi_s^\alpha(\boldsymbol{r})\}$ of $m^{-1}\mathbf{K}$ and the corresponding eigenvalues $1/\lambda$ are defined by

$$\frac{1}{m}\int_D \frac{\mathrm{d}^2 r'}{S} \sum_{s'=\pm} \sum_{\alpha'=1,2} K_{ss'}^{\alpha\alpha'}(\boldsymbol{r}, \boldsymbol{r}')\, \psi_{s'}^{\alpha'}(\boldsymbol{r}') = \frac{1}{\lambda}\, \psi_s^\alpha(\boldsymbol{r}) \tag{3.5}$$

where the domain of integration D is now the strip. Eq. (3.5) corresponds to a set of coupled integral equations

$$\frac{\mathrm{i}\lambda}{4W}\int_D \mathrm{d}^2 r' \left[\frac{1}{\sinh k(\bar{z}-z')}\, \psi_{-s}^1(\boldsymbol{r}') + \frac{1}{\sinh k(\bar{z}-\bar{z}')}\, \psi_{-s}^2(\boldsymbol{r}') \right] = \psi_s^1(\boldsymbol{r}) \tag{3.6a}$$

$$\frac{-\mathrm{i}\lambda}{4W}\int_D \mathrm{d}^2 r' \left[\frac{1}{\sinh k(z-z')}\, \psi_{-s}^1(\boldsymbol{r}') + \frac{1}{\sinh k(z-\bar{z}')}\, \psi_{-s}^2(\boldsymbol{r}') \right] = \psi_s^2(\boldsymbol{r}) \tag{3.6b}$$

for $s = \pm$. In terms of λ,

$$\ln \Xi = \frac{1}{2}\sum_\lambda \ln\left(1 + \frac{m}{\lambda}\right) \tag{3.7}$$

By using the equalities

$$\partial_z \frac{1}{\sinh k(\bar{z}-\bar{z}')} = \partial_{\bar{z}} \frac{1}{\sinh k(z-z')} = \frac{\pi}{k}\, \delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad \boldsymbol{r}, \boldsymbol{r}' \in D \tag{3.8}$$

the integral equations (3.6) can be transformed into the differential equations

$$\partial_z \psi_s^1(\boldsymbol{r}) = \frac{\mathrm{i}\lambda}{2}\, \psi_{-s}^2(\boldsymbol{r}) \tag{3.9a}$$

$$\partial_{\bar{z}} \psi_s^2(\boldsymbol{r}) = -\frac{\mathrm{i}\lambda}{2}\, \psi_{-s}^1(\boldsymbol{r}) \tag{3.9b}$$

The combination of these equations results in the Laplacian eigenvalue problem

$$(-\Delta+\lambda^2)\,\psi_s^\alpha(\mathbf{r})=0 \tag{3.10}$$

with boundary conditions given by the integral equations (3.6):

$$\psi_s^1(x,0)=-\psi_s^2(x,0) \qquad \text{and} \qquad \psi_s^1(x,W)=\psi_s^2(x,W) \tag{3.11}$$

For the present geometry, we look for a solution which is translationally invariant along the x-axis, i.e.,

$$\psi_s^\alpha(x,y)=\mathrm{e}^{ilx}(A_s^\alpha \mathrm{e}^{\kappa y}+B_s^\alpha \mathrm{e}^{-\kappa y}) \tag{3.12}$$

where $\kappa=(l^2+m^2)^{1/2}$. Due to the relations (3.9), only four of eight coefficients $\{A_s^\alpha, B_s^\alpha\}$ are independent, say $\{A_+^\alpha, B_+^\alpha\}$. The boundary conditions (3.11) imply for them a system of four linear homogeneous equations. The existence of a nonvanishing solution is given by the nullity of the determinant of this system. This gives rise to the relation between λ and l:

$$\cosh[W(l^2+\lambda^2)^{1/2}]\pm\lambda\,\frac{\sinh[W(l^2+\lambda^2)^{1/2}]}{(l^2+\lambda^2)^{1/2}}=0 \tag{3.13}$$

The $\pm$ sign means that, for a given l, two λ's with opposite signs occur. Let us define the entire function

$$f(z)=\frac{1}{\cosh(Wl)}\left(\cosh[W(l^2+z^2)^{1/2}]-z\,\frac{\sinh[W(l^2+z^2)^{1/2}]}{(l^2+z^2)^{1/2}}\right) \tag{3.14}$$

The solutions of (3.13) are the zeros of $f(\pm z)$. Since $f(0)=1$, we have

$$f(z)=\prod_{\lambda\in f^{-1}(0)}\left(1-\frac{z}{\lambda}\right) \tag{3.15}$$

Therefore, from (3.7), the grand potential per unit length ω is given by

$$\begin{aligned}\beta\omega&=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}l}{2\pi}\ln\prod_{\lambda\in f^{-1}(0)}\left[\left(1+\frac{m}{\lambda}\right)\left(1-\frac{m}{\lambda}\right)\right]\\&=-\frac{1}{2\pi}\int_0^\infty \mathrm{d}l\,\ln[f(-m)\,f(m)]\end{aligned} \tag{3.16}$$

In the limit $W \to \infty$, it holds

$$\ln f(\pm m) = W[(l^2+m^2)^{1/2} - |l|] + \ln\left(1 \mp \frac{m}{(l^2+m^2)^{1/2}}\right) \\ - \ln(1+e^{-2|l|W}) + O(e^{-mW}) \tag{3.17}$$

Introducing a short-range repulsion cutoff $l_{\max} = 1/\sigma$ in order to avoid the divergence of the bulk term, one has finally, at $\beta = 2$,

$$\beta\omega = -\beta PW + 2\beta\gamma + \frac{\pi}{24W} + O(e^{-mW}) \tag{3.18}$$

where

$$\beta P = \frac{m^2}{2\pi}\left[\ln\left(\frac{2}{m\sigma}\right) + 1\right] \tag{3.19}$$

and

$$\beta\gamma = \frac{m}{4} \tag{3.20}$$

P is the bulk pressure of an infinite system,[3] with a modified cutoff procedure, and γ is the surface tension. The leading finite-size correction term has the universal form $\pi/(24W)$. The same universal term, except for a change of sign, is found for the massless Gaussian field theory defined on the strip, with various boundary conditions of the conformally invariant type.[16]

We notice that the surface tension γ can be computed directly from the density profile obtained in the half-space, formula (2.40). γ is the boundary part per unit length of the grand potential Ω. The total number of particles is given by $N = N_+ + N_- = -\beta\zeta\partial\Omega/\partial\zeta$. The boundary part of this relation is

$$-\beta m \frac{\partial}{\partial m}\gamma = \int_0^\infty dy[n(y) - n] \tag{3.21}$$

where the total particle density $n(y) = n_+(y) + n_-(y)$ and $n = n_+(\infty) + n_-(\infty)$, and we have used that $m = 2\pi\zeta$. With respect to (2.40), it holds

$$\int_0^\infty dy[n(y) - n] = -\frac{m}{4} \tag{3.22}$$

Inserting this into (3.21), one rederives the formula (3.20). This result is also reproduced by the exact solution of the surface tension[1] (valid for an arbitrary $\beta < 3$ and obtained by using completely different means), evaluated at $\beta = 2$.

4. UNIVERSALITY OF THE FINITE-SIZE CORRECTION

The finite-size correction to the grand potential, Eq. (3.18) of the previous section, actually is a special case of a very general result valid at any temperature and in any dimension d ($d \geqslant 2$), for a conducting Coulomb system confined between two parallel ideal dielectric plates separated by a distance W; the Coulomb system extends to infinity in the $d-1$ other directions. We shall show that the grand potential ω per unit area of one plate (times the inverse temperature β) has the large-W expansion

$$\beta\omega = -\beta PW + 2\beta\gamma + \frac{C(d)}{W^{d-1}} + \cdots \tag{4.1}$$

where P is the bulk pressure and γ the surface tension; these quantities are non-universal. However, the last term of (4.1) is a universal finite-size correction, with a coefficient $C(d)$ depending only on the dimension d:

$$C(d) = \frac{\Gamma(d/2)\,\zeta(d)}{2^d \pi^{d/2}} \tag{4.2}$$

where Γ is the Gamma function and ζ the Riemann zeta function. In particular, $C(2) = \pi/24$.

This universal finite-size correction is of the same nature as the ones which occur when the electric potential obeys periodic boundary conditions on the plates[17] or when the plates are ideal conductors (Dirichlet boundary conditions).[7] It is remarkable that $C(d)$ has the same value for ideal conductor and ideal dielectric plates.

In the present case of ideal dielectric plates, two different derivations of (4.1) can be obtained by minor changes in the derivations which have already been made in the case of ideal conductor plates.[7] Here, we shall concentrate on that derivation which relies on the assumption that the Coulomb system exhibits perfect screening properties. Therefore, the universal finite-size correction is *not* expected to hold in the absence of such screening properties, for instance in the low-temperature Kosterlitz–Thouless phase of a two-dimensional Coulomb gas. It should also be noted that if some short-range interactions are added to the Coulomb ones, the screening properties and therefore (4.1) are still expected to hold.

For deriving (4.1) from the screening properties of the Coulomb system, closely following ref. 7, we first establish a sum rule. The d-dimensional Coulomb system is supposed to fill the slab $0<y<W$, with ideal dielectric plates at $y=0$ and $y=W$. Let $\hat{E}_x(0)$ be a Cartesian component parallel to the plates of the microscopic electric field produced by the plasma at some point on the plate $y=0$, say the origin, and let $\hat{\rho}(\mathbf{r})$ be the microscopic charge density at some point in the Coulomb system. If an external infinitesimal dipole p, oriented parallel to the plates, say along the x axis, is placed at the origin (on the Coulomb system side), since the system is assumed to have good screening properties it responds through the appearance of an induced charge density $\delta\rho(\mathbf{r})$, localized near the origin and having a dipole moment opposite to p:

$$\int \mathrm{d}^2 r\; x\delta\rho(\mathbf{r}) = -p \tag{4.3}$$

(choosing p parallel to the plates insures that it has to be screened by the Coulomb system itself, in spite of the presence of images). On the other hand, the interaction Hamiltonian between p and the Coulomb system is $-p\hat{E}_x(0)$ and linear response theory gives $\delta\rho(\mathbf{r}) = \beta p \langle \hat{E}_x(0)\,\hat{\rho}(\mathbf{r})\rangle^T$, where $\langle\cdots\rangle^T$ denotes a truncated two-point function of the unperturbed system. Thus, the correlation function obeys the sum rule

$$\beta \int \mathrm{d}^2 r\; x\langle \hat{E}_x(0)\,\hat{\rho}(\mathbf{r})\rangle^T = -1 \tag{4.4}$$

We now compute the force per unit area acting on the plate $y=0$. Let $\mu_d = (d-2)\,2\pi^{d/2}/\Gamma(d/2)$ if $d>2$, $\mu_2 = 2\pi$. The unit of charge is defined such that the Coulomb interaction between two unit charges in infinite space be $v_0(\mathbf{r}-\mathbf{r}') = |\mathbf{r}-\mathbf{r}'|^{d-2}$ if $d>2$ and $-\ln(|\mathbf{r}-\mathbf{r}'|/a)$ if $d=2$. Then, the Coulomb potential between the ideal dielectric plates can be written as a sum over images

$$v(\mathbf{r},\mathbf{r}') = \sum_{n=-\infty}^{n=\infty} \left[v_0(\mathbf{r}+n2W\mathbf{u}-\mathbf{r}') + v_0(\mathbf{r}^*+n2W\mathbf{u}-\mathbf{r}')\right] \tag{4.5}$$

where $\mathbf{u}$ is the unit vector of the y axis and $\mathbf{r}^* = \mathbf{r}-2y\mathbf{u}$ is an image of $\mathbf{r}$. Actually, the sum in (4.5) does not converge, but it can be made finite through the subtraction of some (infinite) constant which we do not write explicitly since it is irrelevant in what follows. The force per unit area acting on the plate $y=0$ has only a component along the y axis, which is the yy component of the statistical average of the microscopic Maxwell stress tensor

$$T_{yy}(0) = \frac{1}{\mu_d} \langle \hat{E}_y(0)^2 - \frac{1}{2} \hat{\boldsymbol{E}}(0)^2 \rangle = -\frac{(d-1)}{2\mu_d} \langle \hat{E}_x(0)^2 \rangle^T \tag{4.6}$$

where we have used that, on the ideal dielectric plate, $\hat{E}_y(0) = 0$ while all components of $\hat{\boldsymbol{E}}(0)$ parallel to the plate give the same contribution; also, by charge symmetry, the average electric field vanishes and therefore $\langle \cdots \rangle$ can be replaced by $\langle \cdots \rangle^T$. Since the density $n(y)$ vanishes on an ideal dielectric plate (because of the strong particle-image repulsion), the force on the plate has no contact contribution $n(0)\, kT$. Using

$$\hat{E}_x(0) = -\int \mathrm{d}^2 r \left. \frac{\partial v(\boldsymbol{r}, \boldsymbol{r}')}{\partial x'} \right|_{\boldsymbol{r}'=0} \hat{\rho}(\boldsymbol{r}) \tag{4.7}$$

(4.6) can be rewritten as

$$T_{yy}(0) = \frac{(d-1)}{2\mu_d} \int \mathrm{d}^2 r \left. \frac{\partial v(\boldsymbol{r}, \boldsymbol{r}')}{\partial x'} \right|_{\boldsymbol{r}'=0} \langle \hat{E}_x(0)\, \hat{\rho}(\boldsymbol{r}) \rangle^T \tag{4.8}$$

As the distance W between the plates increases, $\partial v / \partial x'$ can be expanded in powers of W^{-1}. Using (4.5) in (4.8) we find

$$T_{yy}(0) = T_{yy}(0)|_{W=\infty} + \frac{(d-1)\, \Gamma(d/2)\, \zeta(d)}{2^d \pi^{d/2} W^d} \int \mathrm{d}^2 r\, x \langle \hat{E}_x(0)\, \hat{\rho}(\boldsymbol{r}) \rangle^T + O\left(\frac{1}{W^{d+1}}\right) \tag{4.9}$$

The integral in (4.9) obeys the sum rule (4.4). Since $\partial \omega / \partial W = T_{yy}(0)$, (4.9) gives (4.1) and (4.2).

In the peculiar case of a one-component plasma, due to the presence of a neutralizing background, the bulk term in the grand potential ω is not of the form $-PW$. Also the force on a plate has an additional term related to the potential difference between the surface and the bulk of the plasma.[18, 19] However that additional term does not contribute to the universal finite-size correction which keeps the same form.[3]

As already noted in ref. 7, the universal finite-size correction does not occur for plain hard walls (dielectric constant $\epsilon = 1$). More generally, our

[3] Although they have been inadvertently omitted in ref. 7, the same remarks apply to the case of ideal conductor plates.

above derivation does not apply to $\epsilon \neq 0, \infty$ because a dipole moment near the wall is not expected to be screened (a phenomenon related to the existence of long-range correlations along the wall). The universal correction holds for field theories with boundary conditions of the conformally invariant type,[16] and this is only in the cases $\epsilon = 0, \infty$ that has been found a mapping of the two-dimensional two-component plasma on such field theories.

5. CONCLUDING REMARKS

The model under consideration was a two-component Coulomb gas in contact with walls made of ideal dielectric material. As shown in Section 2, the model is solvable, in two dimensions at inverse temperature $\beta = 2$, by using the Pfaffian method. This means that, not the grand partition function, but its square is expressible as a "treatable" determinant (an analogous feature was used in the case of the one-dimensional two-component log-gas at the coupling $\beta = 4$[20]). This is the fundamental difference with the previously solved cases of the Coulomb gas in contact with a plain hard wall,[4] or an ideal conductor wall,[4–7] or with periodic boundary conditions on the plates.[17] As a consequence, the introduction of a four-component (instead of two-component) Fermi field, associated with each point of space, is necessary. We have worked with matrices which elements are themselves 2×2 matrices, so without saying it we have used a variant of the algebra of quaternion matrices.

For the rectilinear geometry of a semi-infinite dielectric wall, we have computed the particle densities (2.40) and the correlation functions (2.41). Due to the screening effect, the correlations are supposed to obey a variety of sum rules.[15] In the Appendix, we have checked the monopole (2.42) and quadrupole (2.43) sum rules, with subtraction of (divergent) bulk moments. The relatively complicated form of the truncated pair correlations prevents us, in practice, to go beyond the verification of these sum rules.

The strip formalism in Section 3 is technically very similar to the one for ideal-conductor boundaries.[7] The main formal difference is that, when calculating the grand potential per unit length (3.16), an "average" $(1/2)[\ln f(-m) + \ln f(m)]$ instead of $\ln f(-m)$ should be integrated. This makes the surface tension finite, see formula (3.20). The universal finite-size correction term can also be obtained, as a consequence of the good screening properties, in the more general case of a Coulomb system of arbitrary dimension d ($d \geqslant 2$) confined in a slab of width W, at an arbitrary temperature. The universal finite-size correction has the same value for ideal-conductor and ideal-dielectric walls. In two dimensions, the $\pi/(24W)$ correction term also appears in some papers about the sine-Gordon theory;[21] thus, one might suspect that the integrability of these theories is not a necessary ingredient for obtaining this universal term, and a universal term might also be present in sine-Gordon models of higher dimension although they are not integrable.

APPENDIX

In this Appendix we sketch the derivation of the sum rules (2.42) and (2.43). Without loss of generality, we can choose $x_2 = 0$.

The l.h.s. of (2.42) is

$$F_1 = \left(\frac{m^2}{2\pi}\right)^2 \int_0^\infty \mathrm{d}y_1 \int_{-\infty}^\infty \mathrm{d}x_1 \left\{ K_0(mr_{12})\, K_0(mr_{12}^*) + [K_1(mr_{12}^*)]^2 \right.$$
$$\left. - 2J(x_1, y_1 + y_2)\, x_1 \left[\frac{K_1(mr_{12})}{r_{12}} - \frac{K_1(mr_{12}^*)}{r_{12}^*} \right] \right\} \tag{A1}$$

where $J(x_1, y_1 + y_2)$, as defined by (2.39), has the properties

$$J(0, y_1 + y_2) = 0 \tag{A2}$$

and

$$\frac{\partial J}{\partial x_1} = \int_0^\infty \mathrm{d}l\, \frac{m \cos(l x_1)}{\kappa}\, \mathrm{e}^{-\kappa(y_1+y_2)} = m K_0(mr_{12}^*) \tag{A3}$$

Since

$$m x_1 \left[\frac{K_1(mr_{12})}{r_{12}} - \frac{K_1(mr_{12}^*)}{r_{12}^*} \right] = -\frac{\partial}{\partial x_1} [K_0(mr_{12}) - K_0(mr_{12}^*)] \tag{A4}$$

an integration per partes on x_1 transforms (A1) into

$$F_1 = 2\left(\frac{m^2}{2\pi}\right)^2 \int_0^\infty \mathrm{d}y_1 \int_{-\infty}^\infty \mathrm{d}x_1 \{[K_0(mr_{12}^*)]^2 + [K_1(mr_{12}^*)]^2\} \tag{A5}$$

When K_0 and K_1 in (A5) are replaced by their Fourier transforms with respect to x_1

$$K_0(mr_{12}^*) = \int_{-\infty}^\infty \mathrm{d}l\, \mathrm{e}^{ilx_1} \frac{1}{2\kappa}\, \mathrm{e}^{-\kappa(y_1+y_2)} \tag{A6a}$$

and

$$\frac{\pm \mathrm{i}x_1 + y_1 - y_2}{r_{12}} K_1(mr_{12}^*) = \int_{-\infty}^{\infty} \mathrm{d}l \, \mathrm{e}^{\pm \mathrm{i}lx_1} \frac{l+\kappa}{2m\kappa} \mathrm{e}^{-\kappa(y_1+y_2)} \tag{A6b}$$

(A5) becomes

$$F_1 = \frac{m^2}{2\pi} \int_0^{\infty} \mathrm{d}y_1 \int_{-\infty}^{\infty} \mathrm{d}l \, \mathrm{e}^{-2\kappa(y_1+y_2)} \tag{A7}$$

Performing first the integration on y_1, we obtain

$$F_1 = \frac{m^2}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}l \, \frac{1}{2\kappa} \mathrm{e}^{-2\kappa y_2} = \frac{m^2}{2\pi} K_0(2my_2) \tag{A8}$$

With regard to (2.40), (A8) gives the monopole sum rule (2.42).

The l.h.s. of (2.43) is

$$F_2 = \left(\frac{m^2}{2\pi} \right)^2 \int_0^{\infty} \mathrm{d}y_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 (y_1^2 - x_1^2) \Big\{ K_0(mr_{12}) \, K_0(mr_{12}^*) + [K_1(mr_{12}^*]^2 \\ -2J(x_1, y_1+y_2) \, x_1 \left[\frac{K_1(mr_{12})}{r_{12}} - \frac{K_1(mr_{12}^*)}{r_{12}^*} \right] \Big\} \tag{A9}$$

Now, after an integration per partes on x_1, a J-dependent term is left:

$$F_2 = 2 \left(\frac{m^2}{2\pi} \right)^2 \int_0^{\infty} \mathrm{d}y_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 \Big\{ (y_1^2 - x_1^2)([K_0(mr_{12}^*)]^2 + [K_1(mr_{12}^*)]^2) \\ + \frac{2}{m} J(x_1, y_1+y_2) \, x_1 [K_0(mr_{12}) - K_0(mr_{12}^*)] \Big\} \tag{A10}$$

Again, by the introduction of appropriate Fourier transforms with respect to x_1 (now we also need the Fourier transform of $x_1 K_0(mr_{12})$, etc ...), the integral on x_1 is replaced by an integral on l, and the integration on y_1 can be performed first. After a straightforward but tedious calculation, the detail of which we omit, one obtains

$$F_2 = \frac{m^2}{2\pi} \int_0^{\infty} \mathrm{d}l \left[\frac{m^2 - l^2}{\kappa^4} y_2 + \frac{m^2 - l^2}{\kappa^3} y_2^2 \right] \mathrm{e}^{-2\kappa y_2} \tag{A11}$$

An integration per partes transforms the term $(m^2-l^2)\, y_2/\kappa^4$ into $2l^2y_2^2/\kappa^3$. Thus

$$F_2=y_2^2\,\frac{m^2}{2\pi}\int_0^\infty\frac{\mathrm{d}l}{\kappa}\,\mathrm{e}^{-2\kappa y_2}=y_2^2\,\frac{m^2}{2\pi}\,K_0(2my_2) \tag{A12}$$

With regard to (2.40), (A12) gives the quadrupole sum rule (2.43).

ACKNOWLEDGMENTS

We are grateful to Frédéric van Wijland for very useful discussions. The stay of L.Š. at LPT Orsay is supported by a NATO fellowship. A partial support by Grant VEGA 2/7174/20 is acknowledged.

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